INTERNATIONAL STUDENT
Conference of Economics (ISCE)
University of Mazandaran – 2015, October, 10–11

A Generalization of Initial Conditions in Benchmarking of Economic Time-Series by Additive and Proportional Denton Methods

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Abstract
The paper presents unified analytical solution for combining high-frequency and low-frequency economic time-series by additive and proportional Denton methods with parametrical dependence on the initial values of variable and indicator in evident form. This solution spans Denton's original and Cholette's advanced benchmarking initial conditions as the subcases. Computational complexity of the obtained solution is associated with inversion of a square matrix of the order that is equal to the number of low-frequency observations available. Practical applying the proposed solution under data revisions allows to construct suboptimal concatenation of frozen and newly revised parts of benchmarked time-series by using last benchmarked-to-indicator ratio (or benchmarked and indicator difference in additive case) from the range of data fixed as initial condition for benchmarking or re-benchmarking the newly revised data by the proportional (or additive) Denton method.

Keyword: benchmarking of time-series, movement preservation principle, Denton methods, optimization problem, Lagrange multipliers.

1. Introduction

In the wide sense, benchmarking is a procedure of combining high-frequency and low-frequency economic data for the same flow variable into a consistent time-series. Benchmarking is being implemented, if high-frequency series and low-frequency series do demonstrate the inconsistent movements, in the usual cases when the less frequent data is assumed to be more reliable between two data sets under consideration (see, e.g., [1]).

Let **a** be an ordered set of low-frequency (say, annual) flow data which consists of K elements. Further, let **q** be an ordered T-element set of high-frequency (say, quarterly or monthly) flow data, and let us assume that $T = nK$ where $n \geq 2$ is a certain integer number (frequencies ratio). In other words, n is a number of subperiods (i.e., temporal units for high-frequency data) in one period (i.e., unit time interval for low-frequency data). Thus, one can

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consider benchmark time-series **a** and indicator time-series **q** as two column vectors of dimension $K\times1$ and $T\times1$, respectively.

If these flow time-series show mutually consistent movements, then vectors **q** and **a** must satisfy the following intertemporal condition

$$\mathbf{Sq}=\mathbf{a} \tag{1}$$

where $\mathbf{S}=\mathbf{E}_K\otimes\mathbf{e}'_n$ is the rectangular matrix with dimensions $K\times T$ which can be constructed by *n*-fold successive replication of each column from identity matrix $\mathbf{E}_K$ of order *K*, the character "$\otimes$" denotes the Kronecker product for two matrices, and $\mathbf{e}'_n$ is a transpose of the summation column vector $\mathbf{e}_n$ of dimension $n\times1$ consisting of unit elements.

Otherwise, if the condition (1) is not met, then benchmarking is required to adjust high-frequency time-series **q** for a given benchmark vector **a** to the end that the condition (1) would be satisfied. The adjusted time-series **d** can be represented either in additive form

$$\mathbf{d}=\mathbf{q}+\mathbf{x} \tag{2}$$

or in additive-multiplicative form

$$\mathbf{d}=\mathbf{q}+\hat{\mathbf{q}}\mathbf{y}=\hat{\mathbf{q}}(\mathbf{e}_T+\mathbf{y}) \tag{3}$$

where **x** and **y** are unknown $T\times1$-dimensional vectors of additive and proportional adjustments for forcing less reliable indicator time-series **q** to become more consistent with low-frequency data from **a**. Here putting a "hat" over a vector's symbol (or angled bracketing around it below) denotes a square matrix with the vector on its main diagonal and zeros elsewhere.

Combining (2) and consistency condition (1) gives

$$\mathbf{Sx}=\mathbf{a}-\mathbf{Sq}, \tag{4}$$

whereas combining (3) and (1) leads to

$$\mathbf{S}\hat{\mathbf{q}}\mathbf{y}=\mathbf{a}-\mathbf{Sq}. \tag{5}$$

Thus, a general benchmarking problem is to find additive or proportional adjustments that provide a strict consistency of adjusted time-series **d** and a given benchmark vector **a** in the sense of requirements (4) or (5).

2. The criterion basics for evaluation of the adjustments

It is easy to see that systems of *K* linear equations (4) and (5) contain $T>K$ unknown variables each, so the sets of their feasible solutions are infinite. Hence, a problem of the best choice on the infinite set of feasible adjustments **x** or **y** arises.

The simplest way to determine the feasible adjustments is using of uniform and pro rata distributions. With a uniform distribution for additive case we have

$$\mathbf{x}=\frac{1}{n}(\mathbf{a}-\mathbf{Sq})\otimes\mathbf{e}_n.$$

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Note that inside of any one of K periods the additive adjustments (elements of vector $\mathbf{x}$) coincide with each other. It is easy to check that the condition (4) is met.

In turn, applying a pro rata distribution to additive-multiplicative case (3) leads to

$$\mathbf{e}_T + \mathbf{y} = \langle \mathbf{Sq} \rangle^{-1} \mathbf{a} \otimes \mathbf{e}_n .$$

These proportional adjustments (elements of vector $\mathbf{y}$) also coincide with each other inside of any one of K periods. It can be shown that obtained vector $\mathbf{y}$ satisfies the requirement (5).

However, using of uniform and pro rata distributions is likely to create a discontinuity in the growth rate from the last subperiod of one period to the first subperiod of the next period. This phenomenon is widely known in macroeconomic statistics as so-called “step problem”. Detailed and helpful discussion of the step problem one can find in [1].

A natural way to avoid steps’ appearance in the adjusted high-frequency time-series $\mathbf{d}$ is provided by mathematical programming approach for choice the adjustments that would preserve the short-term movements in high-frequency data as much as possible through period of observations. In the national accounting literature this common notion is usually called “movement preservation principle”. The principle was originally developed by F.Denton [2], who proposed a number of formal representations for movement preservation, including additive and proportional first difference Denton methods, which still are widely used in statistical practice (see [1]). The various aspects of movement preservation principle’s formulation are comprehensively reviewed in [3].

3. Objective functions in additive and proportional Denton methods

For algebraic notation convenience, let us formally expand the vectors $\mathbf{q}$, $\mathbf{d}$, $\mathbf{x}$ and $\mathbf{y}$ by attaching to them the initial scalar values associated with subperiod 0 before the range of observations available. It is important to emphasize that values q_0, d_0, $x_0 = d_0 - q_0$ and $y_0 = d_0 / q_0 - 1$ are unknown a priori.

The objective function of additive first difference Denton method (needed to be minimized) can be written in a form of weighted quadratic variation for time-series $\mathbf{d} - \mathbf{q}$ as

$$\mathrm{f_a}(\mathbf{d}; d_0 - q_0) = w_1 [(d_1 - q_1) - (d_0 - q_0)]^2 + \sum_{t=2}^{T} w_t [(d_t - q_t) - (d_{t-1} - q_{t-1})]^2 \qquad (6)$$

where the weights w_t at $t = 1 \div T$ are the relative reliability (relative confidence) factors for first differences of vector $\mathbf{d} - \mathbf{q}$. Here the character “ $\div$ ” between the lower and upper bounds of index’s changing range means that the index sequentially runs all integer values in the specified range. Clearly, the objective function (6) specifies that the difference between the benchmarked time-series $\mathbf{d}$ and the indicator time-series $\mathbf{q}$ must be as constant as possible through observation period.

Further, the objective function of proportional first difference Denton method (needed to be minimized) can be written as following weighted quadratic variation for element-wise ratio of time-series **d** and **q**:

$$\mathrm{f_p}(\mathbf{d};\, d_0/q_0) = w_1(d_1/q_1 - d_0/q_0)^2 + \sum_{t=2}^{T} w_t(d_t/q_t - d_{t-1}/q_{t-1})^2. \quad (7)$$

Clearly, the objective (7) prescribes the ratio of benchmarked and indicator time-series to be as constant as possible through period of observations.

It is easy to show that putting the additive pattern (2) into formula (6) or the additive-multiplicative pattern (3) into (7) gives the same formal result for both methods as follows:

$$\mathrm{f}\,(\mathbf{z};\, z_0) = w_1(z_1 - z_0)^2 + \sum_{t=2}^{T} w_t(z_t - z_{t-1})^2 \quad (8)$$

where new unknown variable $\mathbf{z}$ and scalar parameter z_0 coincide with $\mathbf{x}$ and x_0 for the additive Denton method and with $\mathbf{y}$ and y_0 for the proportional Denton method, respectively. In new notation the requirements (4) and (5) become

$$\mathbf{Rz} = \mathbf{a} - \mathbf{Sq} \quad (9)$$

where new rectangular $K{\times}T$-dimensional matrix $\mathbf{R}$ coincides with matrix $\mathbf{S}$ for the additive Denton method and with matrix $\mathbf{S\hat{q}}$ for the proportional Denton method.

Thus, the generalized time-series benchmarking problem within Denton first difference approach is to minimize the unified quadratic objective function (8) subject to linear constraints (9). Note that objective function (8) depends on unknown scalar parameter z_0 that is absent in vector equation (9).

4. Initial conditions for benchmarking of time-series

The unified objective function (8) of variable $\mathbf{z}$ with parameter z_0 can be written in matrix notation by two following ways:

$$\mathrm{f}\,(\mathbf{z};\, z_0) = \mathbf{z}'\mathbf{\Delta}'\mathbf{W\Delta\,z} - 2w_1 z_1 z_0 + w_1 z_0^2 \quad (10)$$

and

$$\mathrm{f}\,(\mathbf{z};\, z_0) = \mathbf{z}'\mathbf{\Delta}_1'\mathbf{W}_1\mathbf{\Delta}_1\mathbf{z} + w_1 z_1^2 - 2w_1 z_1 z_0 + w_1 z_0^2 \quad (11)$$

where $\mathbf{W}$ is nonsingular diagonal matrix of order T with weight (relative reliability) coefficients $\mathbf{w} = \{w_t \mid t = 1 \div T\}$ on its main diagonal, and $\mathbf{W}_1$ is its nonsingular square submatrix of order T – 1 obtained by deleting the first column and the first row from matrix $\mathbf{W}$. Usually vector $\mathbf{w}$ is assumed to be normalized by multiplying it on a proper factor, i.e., $\mathbf{e}_T'\mathbf{w} = 1$. Further,

$$\|\boldsymbol{\Delta}\|_{ij}=\begin{cases}1, \text{ when } i=j,\ j=1\div T;\\ -1, \text{ when } i=j+1,\ j=1\div(T-1);\\ 0, \text{ otherwise};\end{cases}\qquad \|\boldsymbol{\Delta}_1\|_{ij}=\begin{cases}-1, \text{ when } i=j,\ j=1\div(T-1);\\ 1, \text{ when } i=j-1,\ j=2\div T;\\ 0, \text{ otherwise};\end{cases}$$

where $\boldsymbol{\Delta}$ is nonsingular left (lower) two-diagonal matrix of order T with units on its main diagonal, and $\boldsymbol{\Delta}_1$ is rectangular two-diagonal matrix with dimensions $(T–1)\times T$ obtained by deleting the first row from matrix Δ. Note that the matrix of quadratic form in right-hand side of (10) is nondegenerate as a product of three invertible matrices. By contrast, the matrix of quadratic form in right-hand side of (11) with the same order T as a product of rectangular matrices with dimensions $T\times(T–1)$ and $(T–1)\times T$ has non-full rank $T – 1$.

Before minimizing the unified objective function $\mathrm{f}\,(\mathbf{z}; z_0)$ subject to linear constraints (9) one needs to make a certain decision concerning a value of unknown parameter z_0. There are three various opportunities in this situation.

First, we can eliminate last two summands in right-hand side of formula (10) by setting z_0 equal to zero. The explanation is as follows: the point $t = 0$ lies outside the observation (and benchmarking) period, so the statement $d_0 = q_0$ seems to be rather acceptable, from which $x_0 = y_0 = z_0 = 0$. (Recall that $x_0 = d_0 - q_0$ and $y_0 = d_0/q_0 - 1$.) This is a main notion of the original postulate developed by F.Denton [2]. Note that Denton initial condition $z_0 = 0$ appears to be quite operational because with it the benchmarking problem becomes a “classical” minimization of *nondegenerate* quadratic form in (10) subject to linear constraints (9). However, using of Denton initial condition generates an artificial bias in estimation of z_1 and launches a spurious transient movement in the early part of benchmarked time-series (for details, see [3]).

Secondly, we can consider unknown parameter z_0 as an additional variable for unconstrained minimization of the unified objective function $\mathrm{f}\,(\mathbf{z}; z_0)$. Setting first partial derivative of the objective function (10) or (11) with respect to z_0 equal to zero gives the simple equation $-2w_1z_1 + 2w_1z_0 = 0$, from which $z_0 = z_1$. This is a main idea of the Denton first difference approach’s modification developed by P.Cholette [4] on a semiempirical background. It is easy to see that Cholette initial condition allows to eliminate last three summands in right-hand side of formula (11). Hence, with this condition the benchmarking problem becomes a constrained minimization of *degenerate* quadratic form in (11) so that a strict identifiability of its variables is achieved due to linear constraints (9) only. Nevertheless, using of Cholette initial condition provides to avoid a main disadvantage of the original Denton condition – a spurious transient movement at the beginning of the time-series, which defeats the movement preservation principle.

Thirdly, we can make an attempt to get an analytical solution of the constrained minimization problem (10), (9) in evident form as a uniparametrical vector family depending on feasible values of z_0.

5. Analytical solution of the minimization problem for benchmarking of time-series

The Lagrangean function for problem to minimize quadratic objective function (10) subject to linear constraints (9) with unknown scalar parameter z_0 is

$$\mathrm{L}(\mathbf{z}; z_0, \boldsymbol{\lambda}) = \mathbf{z}'\boldsymbol{\Delta}'\mathbf{W}\boldsymbol{\Delta}\,\mathbf{z} - 2w_1 z_1 z_0 + w_1 z_0^2 - \boldsymbol{\lambda}'(\mathbf{Rz} + \mathbf{Sq} - \mathbf{a}) \tag{12}$$

where $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers with dimensions $K\times 1$. By setting the partial derivatives of Lagrangean function with respect to $\mathbf{z}$ and $\boldsymbol{\lambda}$ equal to zero for finding its stationary point, we obtain the system of $T+K$ linear equations

$$2\boldsymbol{\Delta}'\mathbf{W}\boldsymbol{\Delta}\,\mathbf{z} - \mathbf{R}'\boldsymbol{\lambda} = 2w_1 z_0 \mathbf{u}, \qquad \mathbf{Rz} = \mathbf{a} - \mathbf{Sq} \tag{13}$$

where $\mathbf{u}$ is the instrumental column vector of dimension $T\times 1$ with unit first element and zeros in all other places.

While symmetric matrix $\mathbf{D} = \boldsymbol{\Delta}'\mathbf{W}\boldsymbol{\Delta}$ of order T is invertible, as noted earlier, the first equation from this system can be resolved with respect to $\mathbf{z}$ as

$$\mathbf{z} = \frac{1}{2}\mathbf{D}^{-1}\mathbf{R}'\boldsymbol{\lambda} + w_1 z_0 \mathbf{D}^{-1}\mathbf{u}.$$

Putting this expression into the second equation of (13) gives a linear estimate of Lagrange multipliers' vector

$$\boldsymbol{\lambda} = 2\left(\mathbf{RD}^{-1}\mathbf{R}'\right)^{-1}\left(\mathbf{a} - \mathbf{Sq} - w_1 z_0 \mathbf{RD}^{-1}\mathbf{u}\right).$$

After backward substitution into the first equation of (13) we obtain following analytical solution of the minimization problem (10), (9) depending on scalar parameter z_0:

$$\mathbf{z}^*(z_0) = \mathbf{D}^{-1}\mathbf{R}'\mathbf{M}^{-1}(\mathbf{a} - \mathbf{Sq}) + w_1 z_0\left(\mathbf{E}_T - \mathbf{D}^{-1}\mathbf{R}'\mathbf{M}^{-1}\mathbf{R}\right)\mathbf{D}^{-1}\mathbf{u} \tag{14}$$

where $\mathbf{M}$ is the symmetric matrix of order K, $\mathbf{M} = \mathbf{RD}^{-1}\mathbf{R}'$, and $\mathbf{E}_T$ denotes an identity matrix of order T.

6. Algebraic transformation of the analytical solution

In obtained analytical solution there are two matrix inverses, namely matrix $\mathbf{D}^{-1}$ of order T and matrix $\mathbf{M}^{-1}$ of order K. The further studying of the algebraic properties for symmetric matrix $\mathbf{D}^{-1} = \boldsymbol{\Delta}^{-1}\mathbf{W}^{-1}(\boldsymbol{\Delta}')^{-1}$ allows to simplify the expression (14).

Clearly, an inverse of the left two-diagonal matrix $\boldsymbol{\Delta}$ of order T is a lower triangular matrix of the same order with unit nonzero elements, whereas an inverse of the right two-diagonal matrix $\boldsymbol{\Delta}'$ is of course an upper triangular matrix. So it is easy to verify (see, e.g., [3])

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that

$$\left\| \boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}')^{-1} \right\|_{ij} = \min\{i, j\}, \quad i = 1 \div T, \ j = 1 \div T.$$

Because $\mathbf{W}^{-1}$ is diagonal matrix with reciprocal weights on its main diagonal, it can be shown that

$$\left\| \mathbf{D}^{-1} \right\|_{ij} = \left\| \boldsymbol{\Delta}^{-1}\mathbf{W}^{-1}(\boldsymbol{\Delta}')^{-1} \right\|_{ij} = \sum_{t=1}^{\min\{i,j\}} \frac{1}{w_t}, \quad i = 1 \div T, \ j = 1 \div T. \tag{15}$$

Thus, the calculations in accordance with analytical solution (14) do not require an inversion of matrix $\mathbf{D}$ with high order $T = nK$ and come to the finding of the inverse $\mathbf{M}^{-1}$ of lower order K. Moreover, formula (15) allows to simplify the right-hand side of expression (14) because it is easy to see that

$$\mathbf{D}^{-1}\mathbf{u} = \frac{1}{w_1}\mathbf{e}_T.$$

Hence, after substitution of latter result, the analytical solution (14) of the minimization problem (10), (9) becomes

$$\mathbf{z}^*(z_0) = \mathbf{D}^{-1}\mathbf{R}'\mathbf{M}^{-1}(\mathbf{a} - \mathbf{Sq}) + z_0\left(\mathbf{e}_T - \mathbf{D}^{-1}\mathbf{R}'\mathbf{M}^{-1}\mathbf{R}\mathbf{e}_T\right). \tag{16}$$

It does not contain instrumental vector $\mathbf{u}$ and demonstrates a linear dependence of additive or proportional benchmarking adjustments $\mathbf{z}$ on initial scalar parameter z_0.

7. An incorporation of Denton initial condition to the analytical solution

The Denton initial condition $z_0 = 0$ provides the most simple representation of the unified analytical solution (16) in a form of vector-valued linear function with the periodic discrepancy vector $\mathbf{a} - \mathbf{Sq}$ as its main argument.

As was noted above, rectangular $K \times T$-dimensional matrix $\mathbf{R}$ coincides with matrix $\mathbf{S}$ for the additive Denton method and with matrix $\mathbf{S}\hat{\mathbf{q}}$ for the proportional Denton method. Hence, $\mathbf{M} = \mathbf{R}\mathbf{D}^{-1}\mathbf{R}' = \mathbf{S}\mathbf{D}^{-1}\mathbf{S}'$ in the first case, and $\mathbf{M} = \mathbf{S}\hat{\mathbf{q}}\mathbf{D}^{-1}\hat{\mathbf{q}}\mathbf{S}'$ in the second case. Thus, under Denton initial condition the analytical solution (16) for additive method and for proportional method can be respectively written as

$$\mathbf{z}_{\mathrm{a}}^*\Big|_{z_0=0} = \mathbf{x}^*\Big|_{x_0=0} = \mathbf{D}^{-1}\mathbf{S}'\left(\mathbf{S}\mathbf{D}^{-1}\mathbf{S}'\right)^{-1}(\mathbf{a} - \mathbf{Sq}), \tag{17}$$

$$\mathbf{z}_{\mathrm{p}}^*\Big|_{z_0=0} = \mathbf{y}^*\Big|_{y_0=0} = \mathbf{D}^{-1}\hat{\mathbf{q}}\mathbf{S}'\left(\mathbf{S}\hat{\mathbf{q}}\mathbf{D}^{-1}\hat{\mathbf{q}}\mathbf{S}'\right)^{-1}(\mathbf{a} - \mathbf{Sq}) \tag{18}$$

where the inverse of matrix $\mathbf{D}$ is determined by formula (15). Note that computational complexity of both Denton methods is equivalent to an inversion of a square matrix with order $K < T$.

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8. An incorporation of Cholette initial condition to the analytical solution

In contrast to Denton initial condition, a handling of the Cholette initial condition $z_0 = z_1$ within the unified analytical solution (16) cannot be implemented directly. However, one can express scalar variable z_1 in terms of z_0 through multiplying both sides of the equation (16) by transpose of $T\times1$-dimensional instrumental vector $\mathbf{u}$ with unit first element and zeros elsewhere, which is introduced earlier in Section 5. For algebraic convenience, let us rewrite the expression (16) as $\mathbf{z}^*(z_0) = \mathbf{g} + z_0\mathbf{h}$ where the vectors $\mathbf{g}$ and $\mathbf{h}$ with dimensions $T\times1$ can be easily determined from the right-hand side of (16).

By definition of vector $\mathbf{u}$ and according to Cholette initial condition, inner product $\mathbf{u}'\mathbf{z}^*(z_0) = z_1$ should be equal to z_0. Thus, we get the scalar equation $z_0 = \mathbf{u}'\mathbf{g} + z_0\mathbf{u}'\mathbf{h}$ with one unknown z_0. Its root at $\mathbf{u}'\mathbf{h} \neq 1$ is

$$z_0 = \frac{\mathbf{u}'\mathbf{g}}{1-\mathbf{u}'\mathbf{h}} \tag{19}$$

where

$$\mathbf{u}'\mathbf{g} = \mathbf{u}'\mathbf{D}^{-1}\mathbf{R}'\mathbf{M}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q}) = (\mathbf{D}^{-1}\mathbf{u})'\mathbf{R}'\mathbf{M}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q}) = \frac{1}{w_t}\mathbf{e}_T'\mathbf{R}'\mathbf{M}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q}),$$

$$\mathbf{u}'\mathbf{h} = \mathbf{u}'(\mathbf{e}_T - \mathbf{D}^{-1}\mathbf{R}'\mathbf{M}^{-1}\mathbf{R}\mathbf{e}_T) = 1-(\mathbf{D}^{-1}\mathbf{u})'\mathbf{R}'\mathbf{M}^{-1}\mathbf{R}\mathbf{e}_T = 1-\frac{1}{w_t}\mathbf{e}_T'\mathbf{R}'\mathbf{M}^{-1}\mathbf{R}\mathbf{e}_T.$$

As a result, substitutions of the latter formulae into right-hand side of (19) give the desired value

$$z_0 = \frac{\mathbf{e}_T'\mathbf{R}'\mathbf{M}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q})}{\mathbf{e}_T'\mathbf{R}'\mathbf{M}^{-1}\mathbf{R}\mathbf{e}_T}, \tag{20}$$

which corresponds to the Cholette initial condition.

For additive Denton method, we have $\mathbf{M}_\mathrm{a} = \mathbf{S}\mathbf{D}^{-1}\mathbf{S}'$ and $\mathbf{R}\mathbf{e}_T = \mathbf{S}\mathbf{e}_T = n\mathbf{e}_K$, whereas the proportional Denton method provides $\mathbf{M}_\mathrm{p} = \mathbf{S}\hat{\mathbf{q}}\mathbf{D}^{-1}\hat{\mathbf{q}}\mathbf{S}'$ and $\mathbf{R}\mathbf{e}_T = \mathbf{S}\hat{\mathbf{q}}\mathbf{e}_T = \mathbf{S}\mathbf{q}$. Thus, under Cholette initial condition $z_0 = z_1$ the analytical solution (16) for additive method and for proportional method becomes

$$\mathbf{z}_\mathrm{a}^*\big|_{z_0=z_1} = \mathbf{x}^*\big|_{x_0=x_1} = \mathbf{D}^{-1}\mathbf{S}'\mathbf{M}_\mathrm{a}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q}) + \frac{\mathbf{e}_K'\mathbf{M}_\mathrm{a}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q})}{n\mathbf{e}_K'\mathbf{M}_\mathrm{a}^{-1}\mathbf{e}_K}(\mathbf{e}_T - n\mathbf{D}^{-1}\mathbf{S}'\mathbf{M}_\mathrm{a}^{-1}\mathbf{e}_K), \tag{21}$$

$$\mathbf{z}_\mathrm{p}^*\big|_{z_0=z_1} = \mathbf{x}^*\big|_{x_0=x_1} = \mathbf{D}^{-1}\hat{\mathbf{q}}\mathbf{S}'\mathbf{M}_\mathrm{p}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q}) + \frac{\mathbf{q}'\mathbf{S}'\mathbf{M}_\mathrm{p}^{-1}(\mathbf{a}-\mathbf{S}\mathbf{q})}{\mathbf{q}'\mathbf{S}'\mathbf{M}_\mathrm{p}^{-1}\mathbf{S}\mathbf{q}}(\mathbf{e}_T - \mathbf{D}^{-1}\hat{\mathbf{q}}\mathbf{S}'\mathbf{M}_\mathrm{p}^{-1}\mathbf{S}\mathbf{q}), \tag{22}$$

respectively, where symmetric matrices $\mathbf{M}_\mathrm{a}$ and $\mathbf{M}_\mathrm{p}$ of order K are defined above, and the inverse of matrix $\mathbf{D}$ is determined by formula (15). Here computational complexity of both methods is also associated with an inversion of a square matrix with order $K < T$, namely matrix $\mathbf{M}_\mathrm{a}$ or $\mathbf{M}_\mathrm{p}$.

9. On idempotency of the considered benchmarking methods

Computational procedures resembling the Denton methods are required to demonstrate an idempotent property to be well-defined. This means that a second applying the estimator (16) with a chosen initial value z_0 should keep invariable the current benchmarked time-series, for which the intertemporal condition (1) holds after first applying.

It is easy to see that both the additive and proportional methods under Denton initial condition satisfy the imposed requirement because the analytical solutions (17), (18) contain (as an efficient) the periodic discrepancy vector $\mathbf{a}-\mathbf{Sq}$ as an efficient, which becomes a null (zero) vector after the first run of a method.

In the case of Cholette initial condition one can see a similar picture. The first summands and also the parameter values z_0 in analytical solutions (21), (22) depend on the periodic discrepancy vector $\mathbf{a}-\mathbf{Sq}$. Hence, the additive and proportional Denton methods with Cholette initial condition are idempotent too.

From brief analysis above, it becomes clear that an arbitrary choice of parameter value z_0 in unified estimator (16) can violate an idempotent property of a method. The most general approach to keep on an idempotency is to choose the parameter z_0 as a scalar function of vector $\mathbf{a}-\mathbf{Sq}$ that vanishes at $\mathbf{a}=\mathbf{Sq}$. Moreover, the most operational way is to restrict an existing variety of these function by the linear ones, i.e., to set

$$z_0 = \boldsymbol{\beta}'(\mathbf{a}-\mathbf{Sq}) \tag{23}$$

where $\boldsymbol{\beta}'$ is a transpose of the certain column vector with dimensions $K\times 1$ that should be specified in advance.

Function family (23) depicts a wide class of the modes for constructing the recursive base of additive and proportional Denton methods. Indeed, if vector $\boldsymbol{\beta}$ is located on a hyperplane that is orthogonal to the periodic discrepancy vector $\mathbf{a}-\mathbf{Sq}$ given, then $z_0 = 0$ as in the Denton initial condition. Further, by setting $\boldsymbol{\beta}' = \mathbf{e}'_T\mathbf{R}'\mathbf{M}^{-1} / \mathbf{e}'_T\mathbf{R}'\mathbf{M}^{-1}\mathbf{R}\mathbf{e}_T$ we obtain the formula (20) for the Cholette initial condition.

It is appropriate mention here that the other specifications of an instrumental vector $\boldsymbol{\beta}$ generate a variety of initial conditions for flexible implementation of a benchmarking method

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subject to the requirements of its idempotency. It appears that this issue deserves further study.

10. Practical applying the proposed benchmarking solutions under the data revisions

One of the main questions concerning a practical use of Denton methods is how to apply them correctly in the circumstances of the recent data revisions and the new data arrivals. Clearly, the revisions of the already benchmarked time-series for a few recent time periods generally violate the intertemporal condition (1) within these periods. As for new data just arrived, it is needed to be benchmarked in turn while keeping invariable the part of time-series that was benchmarked (and published) earlier.

In the relevant literature devoted to the Denton and other benchmarking methods one can find some misty recommendations as folows. “To avoid introducing distortions in the series, incorporation of new annual data for one year will generally require revision of previously published quarterly data for several years. This is a basic feature of all acceptable benchmarking methods” (see [1], par. 6.49). “When new data become available, benchmarking must be applied to all the available data, namely the past un-benchmarked series and the corresponding benchmarks, followed by the current un-benchmarked series and the newly available benchmark(s). Benchmarking must never be applied to an already benchmarked series followed by a segment of new data” (see [3], p.10).

From the quotations above, it is not quite clear what kind of the benchmarking initial conditions is better to choose and how it should be applied through the calculations. However, the proposed generalization of initial conditions in a form of the unified analytical solution (16) allows to give an explicit answer to this practically important question.

Let us consider a dichotomy of the observation period between a range of fixed (say, already published) data $t = 1 \div \tau$ and a range of revising and/or new data $t = (\tau+1) \div T$. The data from the first range are transformed earlier by a certain benchmarking method, so the indicator value q_τ and the benchmarked value d_τ in a border point $t = \tau$ are already known as well as the value of parameter z_τ which is equal to $d_\tau - q_\tau$ for additive Denton method and $d_\tau / q_\tau - 1$ for proportional Denton method. Applying the estimator (16) with $z_0 = z_\tau$ to the revising and/or new data set at $t = (\tau+1) \div T$ allows to smoothly concatenate both parts of (now fully benchmarked) high-frequency time-series under consideration.

A reasonable “pay” for the dichotomy is, of course, some increase of the quadratic variation (8) for the concatenated time-series in comparison with time-series optimally benchmarked as a whole. Besides, note that an idempotency of this two-stage algorithm is provided by idempotency of the mentioned above “certain benchmarking method”, which is used on the first stage of the algorithm. Its second applying leads to $z_\tau = 0$, and the estimator (16) used on the second stage of the algorithm becomes idempotent.

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11. Lagrange multipliers and the solution sensitivity analysis

It is assumed until now that the linear vector constraint (9) is binding, i.e., it should hold strictly. An impact following a refusal of the binding equality constraint can be measured by means of a technique of sensitivity analysis widely used in mathematical programming.

From the optimization theory is known that Lagrange multipliers in the optimal solution of a mathematical programming problem with equality constraints are the components of the objective function's gradient with respect to the right-hand sides of constraints at optimum point – for example, see [5], pp. 160, 161. Note that in our study the right-hand side of constraint (9) are expressed in terms of the periodic discrepancy vector $\mathbf{a}-\mathbf{Sq}$, and, moreover, for the proportional Denton method the left-hand side of constraint (9) also depends on indicator time-series $\mathbf{q}$ because of $\mathbf{R}=\mathbf{S}\hat{\mathbf{q}}$ in this case. Thus, by Taylor's expansion in a neighborhood of minimum point $\mathbf{z}^*(z_0)$ we have

$$\partial\, \mathrm{f}\,(\mathbf{z}^*; z_0) = -\boldsymbol{\lambda}'\partial\left(\mathbf{Rz}^* + \mathbf{Sq} - \mathbf{a}\right) = \boldsymbol{\lambda}'\partial\mathbf{a} - \boldsymbol{\lambda}'\left(\frac{\partial\mathbf{Rz}^*}{\partial\mathbf{q}} + \mathbf{S}\right)\partial\mathbf{q}.$$

Therefore, the partial derivatives of the objective function (10) with respect to $\mathbf{a}$ and $\mathbf{q}$ at its minimum point are expressed as follows:

$$\frac{\partial\, \mathrm{f}\,(\mathbf{z}^*; z_0)}{\partial\mathbf{a}} = \boldsymbol{\lambda}, \qquad \frac{\partial\, \mathrm{f}\,(\mathbf{z}^*; z_0)}{\partial\mathbf{q}} = -\left(\frac{\partial(\mathbf{Rz}^*)'}{\partial\mathbf{q}} + \mathbf{S}'\right)\boldsymbol{\lambda}.$$

It is easy to see that for additive Denton method the partial derivative in the right-hand side vanishes, i.e.

$$\frac{\partial(\mathbf{Rz}^*)'}{\partial\mathbf{q}} = \frac{\partial(\mathbf{Sz}^*)'}{\partial\mathbf{q}} = \mathbf{0}_T$$

where $\mathbf{0}_T$ is a $T\times 1$-dimensional zero vector. Further, for proportional Denton method the latter formula can be rewritten as

$$\frac{\partial(\mathbf{Rz}^*)'}{\partial\mathbf{q}} = \frac{\partial(\mathbf{S}\hat{\mathbf{q}}\mathbf{z}^*)'}{\partial\mathbf{q}} = \frac{\partial(\mathbf{S}\hat{\mathbf{z}}^*\mathbf{q})'}{\partial\mathbf{q}} = \hat{\mathbf{z}}^*\mathbf{S}'$$

where an obvious commutativity property of diagonal matrices is used.

Finally, the partial derivatives of the objective function (10) with respect to $\mathbf{a}$ and $\mathbf{q}$ for additive and proportional Denton methods respectively become

$$\frac{\partial\, \mathrm{f}_{\mathrm{a}}\,(\mathbf{z}_{\mathrm{a}}^*; z_0)}{\partial\mathbf{a}} = \boldsymbol{\lambda}_{\mathrm{a}}, \qquad \frac{\partial\, \mathrm{f}_{\mathrm{a}}\,(\mathbf{z}_{\mathrm{a}}^*; z_0)}{\partial\mathbf{q}} = -\,\mathbf{S}'\boldsymbol{\lambda}_{\mathrm{a}}; \tag{24}$$

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$$\frac{\partial \mathrm{f}_{\mathrm{p}}(\mathbf{z}_{\mathrm{p}}^{*};z_0)}{\partial \mathbf{a}}=\boldsymbol{\lambda}_{\mathrm{p}}, \qquad \frac{\partial \mathrm{f}_{\mathrm{p}}(\mathbf{z}_{\mathrm{p}}^{*};z_0)}{\partial \mathbf{q}}=-\left(\mathbf{E}_T+\hat{\mathbf{z}}_{\mathrm{p}}^{*}\right)\mathbf{S}'\boldsymbol{\lambda}_{\mathrm{p}} \qquad (25)$$

where $\mathbf{E}_T$ is an identity matrix of order T.

The vector of Lagrange multipliers for problem to minimize quadratic objective function (10) subject to linear constraints (9) with given scalar parameter z_0 is derived like in Section 5. After the algebraic transformation considered in Section 6 this vector becomes

$$\boldsymbol{\lambda}=2\mathbf{M}^{-1}\left(\mathbf{a}-\mathbf{Sq}\right)-2z_0\,\mathbf{M}^{-1}\mathbf{Re}_T$$

where $\mathbf{M}=\mathbf{RD}^{-1}\mathbf{R}'$, as earlier.

In particular, for the additive Denton method we have $\mathbf{Re}_T=\mathbf{Se}_T=n\mathbf{e}_K$, and so

$$\boldsymbol{\lambda}_{\mathrm{a}}=2\mathbf{M}^{-1}\left(\mathbf{a}-\mathbf{Sq}\right)-2nz_0\,\mathbf{M}^{-1}\mathbf{e}_K. \qquad (26)$$

The proportional Denton method gives $\mathbf{Re}_T=\mathbf{S}\hat{\mathbf{q}}\mathbf{e}_T=\mathbf{Sq}$, hence,

$$\boldsymbol{\lambda}_{\mathrm{p}}=2\mathbf{M}^{-1}\left(\mathbf{a}-\mathbf{Sq}\right)-2z_0\,\mathbf{M}^{-1}\mathbf{Sq}. \qquad (27)$$

It is important to emphasize that Lagrange multipliers (26) and (27) should be used in conjunction with two gradients (24) for additive Denton method and other two gradients (25) for proportional Denton method, respectively.

A small shift in a space of vectors **a** or **q** along the antigradient defined by reversing a sign in (24) or (25) entails a small decrease of the quadratic variation (8) in comparison with its optimal value determined originally. In this context the larger absolute values of antigradient's elements are of great interest. Such sensitivity analysis allows to detect the elements of initial benchmark time-series **a** and indicator time-series **q** that serves as main sources of the movement preservation principle's violation.

12. Concluding remarks

The unified analytical solution (16) demonstrates a high degree of flexibility while implementing the benchmarking calculations in statistical practice. Adaptive features of the developed approach are provided, first, by using a set of the relative reliability (weight) coefficients, secondly, by means of combining the additive and proportional Denton methods into a united algorithm, and, thirdly, by a flexible choice of the initial conditions within Denton's benchmarking framework based on a movement preservation principle.

Practical applying the proposed analytical solution in the circumstances of the recent data revisions and the new data arrivals allows to construct suboptimal concatenation of the fixed part and the newly revised part of benchmarked time-series by using the last value of scalar parameter z_0 in estimator (16) from the range of data frozen as initial condition for benchmarking or re-benchmarking the newly revised data by the additive or proportional Denton method.

Computing efficiency of the developed tools is quite high because associated calculations come to inversion of a square matrix of the order K that equals a number of observations in low-frequency data set available.

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